

## ELEMENTARY MAGMA GRADINGS ON RINGS

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ABSTRACT. Suppose that  $G$  and  $H$  are magmas and that  $R$  is a strongly  $G$ -graded ring. We show that there is a bijection between the set of elementary (nonzero)  $H$ -gradings of  $R$  and the set of (zero) magma homomorphisms from  $G$  to  $H$ . Thereby we generalize a result by Dăscălescu, Năstăsescu and Rios Montes from group gradings of matrix rings to strongly magma graded rings. We also show that there is an isomorphism between the preordered set of elementary (nonzero)  $H$ -filters on  $R$  and the preordered set of (zero) submagmas of  $G \times H$ . These results are applied to category graded rings and, in particular, to the case when  $G$  and  $H$  are groupoids. In the latter case, we use this bijection to determine the cardinality of the set of elementary  $H$ -gradings on  $R$ .

## 1. INTRODUCTION

A classical problem in ring theory is to find all group gradings on matrix rings (see e.g. [3], [4], [5], [6], [7], [8], [13], [17], [18], [25]). More precisely, if  $H$  is a group,  $n$  is a positive integer,  $K$  is a field and  $M_n(K)$  denotes the ring of  $n \times n$  matrices with entries in  $K$ , then an  $H$ -grading on  $M_n(K)$  is a collection of  $K$ -subspaces  $(W_h)_{h \in H}$  of  $M_n(K)$  satisfying  $M_n(K) = \bigoplus_{h \in H} W_h$  and  $W_h W_{h'} \subseteq W_{hh'}$  for all  $h, h' \in H$ . In [8] Dăscălescu, Năstăsescu and Rios Montes determine all  $H$ -gradings on  $M_n(K)$  in the case when  $H$  is a finite cyclic group with  $q$  elements and  $K$  contains a primitive  $q$ th root of 1. In loc. cit. Dăscălescu et. al. also determine all cyclic gradings of order two on  $M_2(K)$  for any field  $K$ . In [3] Bahturin and Sehgal describe all  $H$ -gradings on  $M_n(K)$  when  $H$  is abelian and  $K$  is an algebraically closed field. The general case is still unsettled. However, if we restrict ourselves to the problem of finding all *elementary*  $H$ -gradings on  $M_n(K)$ , then there is a complete answer to this problem (see Theorem 1). Namely, following the notation introduced in [3], an  $H$ -grading on  $M_n(K)$  is called elementary if each  $W_h$ , for  $h \in H$ , considered as a left vector space over  $K$ , has a basis consisting of matrices of the type  $e_{i,j}$  with 1 at the  $(i,j)$  position and 0 elsewhere. Note that the concept of elementary gradings of matrix algebras have appeared in another setting in the work of E. L. Green and E. N. Marcos (see [10] and [11]) where they view  $M_n(K)$  as a quotient of the path algebra of the complete graph on  $n$  points, the elementary gradings arise from weight functions on this graph.

**Theorem 1** (Dăscălescu, Năstăsescu and Rios Montes [8]). *If  $H$  is a finite group with  $q$  elements, then there are  $q^{n-1}$  elementary  $H$ -gradings on the matrix algebra  $M_n(K)$ .*

The main purpose of this article is to generalize Theorem 1 from group gradings on matrix algebras to groupoid gradings on groupoid algebras (see Theorem 2). The impetus for this generalization is the observation that  $M_n(K)$  is a groupoid algebra over  $K$ . Namely, suppose that  $\Gamma$  is a category. The objects and morphisms in  $\Gamma$  are denoted by  $\text{ob}(\Gamma)$  and  $\text{mor}(\Gamma)$  respectively. The domain and codomain of  $s \in \text{mor}(\Gamma)$  are denoted  $d(s)$  and  $c(s)$  respectively. If  $e \in \text{ob}(\Gamma)$ , then we let  $\Gamma_e$  denote the monoid of all morphisms from  $e$  to  $e$ . In particular, we let  $\text{id}_e$  denote the identity morphism  $e \rightarrow e$ . If  $e, e' \in \text{ob}(\Gamma)$ , then we let  $\text{hom}(e, e')$  denote the set of morphisms  $e \rightarrow e'$ . Recall that  $\Gamma$  is called a *groupoid* if all of its morphisms are isomorphisms. Note that if  $\Gamma$  is a groupoid, then  $\Gamma_e$  is a group for all  $e \in \text{ob}(\Gamma)$ . A category  $\Gamma$  is called *thin* (*connected*) if for all  $x, y \in \text{ob}(\Gamma)$ , there is at most (at least) one morphism from  $x$  to  $y$ . A category is called *finite* if the set of morphisms of the category is finite. Recall that the category algebra of  $\Gamma$  over  $K$ , denoted  $K[\Gamma]$ , is the collection of formal sums  $\sum_{s \in \text{mor}(\Gamma)} a_s s$ , where  $a_s \in K$ , for  $s \in \text{mor}(\Gamma)$ , are chosen so that all but finitely many of them are nonzero. The addition and multiplication on  $K[\Gamma]$  is defined by  $\sum_{s \in \text{mor}(\Gamma)} a_s s + \sum_{s \in \text{mor}(\Gamma)} b_s s = \sum_{s \in \text{mor}(\Gamma)} (a_s + b_s) s$  and  $\sum_{s \in \text{mor}(\Gamma)} a_s s \sum_{t \in \text{mor}(\Gamma)} b_t t = \sum_{r \in \text{mor}(\Gamma)} \sum_{d(s)=c(t), st=r} a_s b_t r$  respectively for all  $\sum_{s \in \text{mor}(\Gamma)} a_s s \in K[\Gamma]$  and all  $\sum_{t \in \text{mor}(\Gamma)} b_t t \in K[\Gamma]$ . If  $\Gamma$  is a groupoid, a monoid or a group, then  $K[\Gamma]$  is called a groupoid algebra, a monoid algebra or a group algebra respectively. Note that if  $\Gamma$  is a connected groupoid, then the cardinality of  $\text{hom}(\Gamma_e, \Gamma_{e'})$  is the same for all choices of  $e, e' \in \text{ob}(\Gamma)$ . If  $\Lambda$  also is a connected groupoid, then the cardinality of  $\text{hom}(\Gamma_e, \Lambda_f)$  is the same for all choices of  $e \in \text{ob}(\Gamma)$  and  $f \in \text{ob}(\Lambda)$ . For the details concerning these claims, see the discussion in the end of Section 3 where we show the following result.

**Theorem 2.** *Let  $\Gamma$  and  $\Lambda$  be finite connected groupoids and take  $e \in \text{ob}(\Gamma)$  and  $f \in \text{ob}(\Lambda)$ . If we put  $m = |\text{ob}(\Gamma)|$ ,  $n = |\text{ob}(\Lambda)|$ ,  $p = |\text{hom}(\Gamma_e, \Lambda_f)|$  and  $q = |\text{hom}(\Gamma_e, \Gamma_e)|$ , then there are  $(pq^{m-1})^{n^m}$  elementary  $\Lambda$ -gradings on the groupoid algebra  $K[\Gamma]$ .*

Theorem 2 generalizes Theorem 1. Indeed, if we let  $\Gamma$  be the unique thin connected groupoid with  $n$  objects, that is, if we let  $\Gamma$  have the first  $n$  positive integers as objects and the matrices  $e_{i,j}$  as morphisms, where  $d(e_{i,j}) = j$  and  $c(e_{i,j}) = i$ , then  $M_n(K)$  equals the groupoid algebra  $K[\Gamma]$ . If we now let  $\Lambda$  be a group, that is a groupoid with one object, then, by Theorem 2, it follows that there are  $(pq^{m-1})^{n^m} = (1 \cdot q^{n-1})^{1^m} = q^{n-1}$  elementary  $\Lambda$ -gradings on  $K[\Gamma]$ . For the relevant definitions concerning elementary category gradings on rings, see Section 3. Note that one may, using Theorem 2,

count the cardinality of the set of elementary  $\Lambda$ -gradings of  $K[\Gamma]$  for any (not necessarily connected) finite groupoids  $\Gamma$  and  $\Lambda$  (see Remark 5 in Section 3).

Theorem 1 is in [8] shown by displaying an explicit bijection between the set of elementary  $H$ -gradings on  $M_n(K)$  and the set  $H^{n-1}$ . This bijection is defined in three steps. Namely, suppose that we are given  $h_i \in H$ , for  $i = 1, 2, \dots, n-1$ . Step 1: If  $1 \leq i \leq n-1$ , then map  $e_{i,i+1}$  to  $h_i$ . Step 2: If  $1 \leq i < j \leq n$ , then map  $e_{i,j}$  to  $h_i h_{i+1} \cdots h_{j-1}$ . Step 3: If  $1 \leq j \leq i \leq n$ , then map  $e_{i,j}$  to  $h_{i-1}^{-1} h_{i-2}^{-1} \cdots h_j^{-1}$ . In Section 3, we generalize this bijection to category graded and filtered rings (see Theorem 5 and Theorem 6). In fact, we even show that the category graded bijection follows from the following general result that holds for certain magma graded rings.

**Theorem 3.** *If  $G$  and  $H$  are (zero) magmas and  $R$  is a ring equipped with a nonzero strong  $G$ -grading, then there is a bijection between the set of elementary (nonzero)  $H$ -gradings on  $R$  and the set of (zero) magma homomorphisms from  $G$  to  $H$ .*

For a proof of Theorem 3 and the relevant definitions concerning (zero) magmas and magma gradings on rings, see Section 2.

In this article, we also introduce and analyse the problem of finding all elementary magma *filters* on a given ring. For the definition of this concept, see Section 2. Note that magma filters on rings have been extensively studied in the case when the magma equals the nonnegative integers equipped with addition as operation. Namely, in that case a filter on a ring is a collection  $(W_n)_{n \geq 0}$  of additive subgroups of the ring satisfying  $W_n W_{n'} \subseteq W_{n+n'}$  for all nonnegative integers  $n$  and  $n'$ . For more details concerning this, see any standard book on commutative ring theory, e.g. Chapter III in [2]. We remark that in the literature there do not seem to exist any results concerning the problem of finding all magma filters on rings - not even in the case when the magma equals the nonnegative integers. In Section 2, we show the following result.

**Theorem 4.** *If  $G$  and  $H$  are (zero) magmas and  $R$  is a ring equipped with a nonzero strong  $G$ -grading, then there is an isomorphism between the pre-ordered set of elementary (nonzero)  $H$ -filters on  $R$  and the preordered set of (zero) submagmas of  $G \times H$ .*

In the end of Section 2, we apply Theorem 3 and Theorem 4 on magma rings over  $K$  (see Corollary 1 and Corollary 2). In particular, we use these bijections to determine the number of magma gradings and magma filters on rings in some concrete cases (see Examples 1-6). In Section 3, we apply Theorem 3 and Theorem 4 on category graded and category filtered rings (see Theorem 5 and Theorem 6) and exemplify this in some concrete cases (see Examples 7-11). In the end of this section, we show Theorem 2.

## 2. MAGMA FILTERED RINGS

In this section, we recall some fairly well known notions from the theory of magmas (see Definition 1). For more details concerning this, see e.g. the book [13] by Kelarev (where however the notion of magma, found e.g. in [2], is called groupoid). We also introduce the concept of (elementary) magma filtered rings (see Definition 2). Then we show Theorem 3 and Theorem 4 through a series of results (see Propositions 1-4) some of which hold in a more general context. We also introduce the notions of zero magma and zero magma homomorphism (see Definition 3). In the end of this section, we apply Theorem 3 and Theorem 4 to several different cases of magma algebras (see Corollary 1, Corollary 2 and Examples 1-6).

**Definition 1.** For the rest of the section, let  $G$  be a magma. By this we mean that  $G$  is a set equipped with a binary operation  $G \times G \ni (g, g') \mapsto gg' \in G$ . We consider the empty set to be a magma. By a submagma of  $G$  we mean a subset of  $G$  which is closed under the binary operation on  $G$ . Let  $H$  be another magma. The product set  $G \times H$  has a natural structure of a magma induced by the binary operations on  $G$  and  $H$ . If  $f$  is a subset of  $G \times H$  and  $h \in H$ , then we let  $f^{-1}(h)$  denote the collection of  $g \in G$  such that  $(g, h) \in f$ . We let  $S(G \times H)$  denote the partially ordered set of submagmas of  $G \times H$  ordered by inclusion. We say that a function  $f : G \rightarrow H$  is a homomorphism of magmas if  $f(gg') = f(g)f(g')$  for all  $g, g' \in G$ . We let  $\text{hom}(G, H)$  denote the set of magma homomorphisms from  $G$  to  $H$ .

**Definition 2.** For the rest of the section, let  $R$  be an associative ring equipped with a  $G$ -filter  $(V_g)_{g \in G}$ . By this we mean that  $(V_g)_{g \in G}$  is a collection of additive subgroups of  $R$  satisfying  $V_g V_{g'} \subseteq V_{gg'}$  for all  $g, g' \in G$ . We say that such a  $G$ -filter is *nonzero (strong)* if  $V_g \neq \{0\}$  for all  $g \in G$  ( $V_g V_{g'} = V_{gg'}$  for all  $g, g' \in G$ ). Furthermore, we say that an  $H$ -filter  $(W_h)_{h \in H}$  on  $R$  is *elementary* (with respect to  $(V_g)_{g \in G}$ ) if  $W_h = \sum_{V_g \subseteq W_h} V_g$  for all  $h \in H$ . We let  $G(R)$  denote the partially ordered set of  $G$ -filters on  $R$ , the ordering defined by saying that  $(V_g)_{g \in G} \leq (W_g)_{g \in G}$  if  $V_g \subseteq W_g$  for all  $g \in G$ . We say that the  $G$ -filter  $(V_g)_{g \in G}$  on  $R$  is a  $G$ -grading if  $R = \bigoplus_{g \in G} V_g$ . A  $G$ -grading on  $R$  is called nonzero (strong) if it is nonzero (strong) as a  $G$ -filter.

**Proposition 1.** *The function  $F : S(G \times H) \rightarrow H(R)$  defined by  $F(f)_h = \sum_{g \in f^{-1}(h)} V_g$ , for  $h \in H$  and  $f \in S(G \times H)$ , is a homomorphism of partially ordered sets. If  $(V_g)_{g \in G}$  is a grading on  $R$  and  $f : G \rightarrow H$  is a homomorphism of magmas, then  $F(f)$  is an  $H$ -grading on  $R$ .*

*Proof.* First we show that  $F$  is well defined. Take  $h, h' \in H$  and a submagma  $f$  of  $G \times H$ . Then

$$F(f)_h F(f)_{h'} = \sum_{\substack{g \in f^{-1}(h) \\ g' \in f^{-1}(h')}} V_g V_{g'} \subseteq \sum_{\substack{g \in f^{-1}(h) \\ g' \in f^{-1}(h')}} V_{gg'} \subseteq$$

$$\subseteq \sum_{g'' \in f^{-1}(hh')} V_{g''} = F(f)_{hh'}.$$

Therefore  $F(f)$  is an  $H$ -filter on  $R$ . Now we show that  $F$  respects inclusion. Suppose that  $f'$  is another submagma of  $G \times H$  such that  $f \subseteq f'$ . Then

$$F(f)_h = \sum_{g \in f^{-1}(h)} V_g \subseteq \sum_{g \in f'^{-1}(h)} V_g = F(f')_h.$$

Therefore  $F(f) \leq F(f')$ . If  $f : G \rightarrow H$  is a homomorphism of magmas and  $(V_g)_{g \in G}$  is a  $G$ -grading on  $R$ , then the sets  $f^{-1}(h)$ , for  $h \in H$ , are pairwise disjoint and cover  $G$ . Therefore, we get that  $R = \bigoplus_{g \in G} V_g = \bigoplus_{h \in H} (\bigoplus_{g \in f^{-1}(h)} V_g) = \bigoplus_{h \in H} F(f)_h$ .  $\square$

**Proposition 2.** *Suppose that the  $G$ -filter  $(V_g)_{g \in G}$  on  $R$  is strong. Then the function  $M : H(R) \rightarrow S(G \times H)$  defined by saying that  $(g, h) \in M((W_h)_{h \in H})$ , for an  $H$ -filter  $(W_h)_{h \in H}$  on  $R$ , precisely when  $V_g \subseteq W_h$ , is a homomorphism of partially ordered sets. If  $(V_g)_{g \in G}$  is nonzero,  $(W_h)_{h \in H}$  is a grading on  $R$  and there to each  $g \in G$  is  $h \in H$  with  $V_g \subseteq W_h$ , then  $M((W_h)_{h \in H})$  is a homomorphism of magmas.*

*Proof.* First we show that  $M$  is well defined. Suppose that  $(W_h)_{h \in H}$  is an  $H$ -filter on  $R$  and that  $(g, h)$  and  $(g', h')$  belong to  $M((W_h)_{h \in H})$ . Then  $V_g \subseteq W_h$  and  $V_{g'} \subseteq W_{h'}$ . Hence, since the  $G$ -filter  $(V_g)_{g \in G}$  is strong, we get that  $V_{gg'} = V_g V_{g'} \subseteq W_h W_{h'} \subseteq W_{hh'}$ . Therefore  $(gg', hh') \in M((W_h)_{h \in H})$ .

Now we show that  $M$  respects the partial orders. Suppose that  $(W'_h)_{h \in H}$  is another  $H$ -filter on  $R$  such that  $(W_h)_{h \in H} \leq (W'_h)_{h \in H}$ . Take  $(g, h) \in M((W_h)_{h \in H})$ . Then  $V_g \subseteq W_h \subseteq W'_h$  and hence  $(g, h) \in M((W'_h)_{h \in H})$ . Therefore we get that  $M((W_h)_{h \in H}) \subseteq M((W'_h)_{h \in H})$ .

Now suppose that  $(V_g)_{g \in G}$  is nonzero,  $(W_h)_{h \in H}$  is an  $H$ -grading on  $R$ , and there to each  $g \in G$  is  $h \in H$  with  $V_g \subseteq W_h$ . We show is that  $M((W_h)_{h \in H})$  is a function. Seeking a contradiction, suppose that there are  $h, h' \in H$  with  $h \neq h'$  and  $V_g \subseteq W_h$  and  $V_g \subseteq W_{h'}$ . Then, since  $(W_h)_{h \in H}$  is an  $H$ -grading on  $R$ , we get that  $\{0\} \subsetneq V_g \subseteq W_h \cap W_{h'} = \{0\}$  which is a contradiction.  $\square$

**Proposition 3.** *If  $f \in S(G \times H)$ , then  $M(F(f)) \supseteq f$  with equality if  $(V_g)_{g \in G}$  is a nonzero strong  $G$ -grading on  $R$ .*

*Proof.* Take  $f \in S(G \times H)$ . Then  $M(F(f)) = \{(g, h) \in G \times H \mid V_g \subseteq \sum_{g' \in f^{-1}(h)} V_{g'}\} \supseteq f$ . Now suppose that  $(V_g)_{g \in G}$  is a nonzero strong  $G$ -grading on  $R$ . Take  $(g, h) \in G \times H$  such that  $V_g \subseteq \sum_{g' \in f^{-1}(h)} V_{g'}$ . Then  $V_g = V_{g'}$  for some  $g' \in f^{-1}(h)$ . But this implies that  $g = g' \in f^{-1}(h)$  and hence  $(g, h) \in f$ . Therefore  $M(F(f)) \subseteq f$ .  $\square$

**Proposition 4.** *If  $(V_g)_{g \in G}$  is a strong  $G$ -filter on  $R$  and  $(W_h)_{h \in H}$  is an elementary  $H$ -filter on  $R$ , then  $F(M((W_h)_{h \in H})) = (W_h)_{h \in H}$ .*

*Proof.* Take  $h' \in H$ . Then  $F(M((W_h)_{h \in H}))_{h'} = \sum_{g \in M((W_h)_{h \in H})^{-1}(h')} V_g = \sum_{V_g \subseteq W_{h'}} V_g = W_{h'}$ .  $\square$

**Definition 3.** Let  $G$  be a *zero magma*. By this we mean that  $G$  is equipped with a *zero element*, that is an element  $0$  satisfying  $0g = g0 = 0$  for all  $g \in G$ . Let  $H$  be another zero magma. A subset  $f$  of  $G \times H$  is called a *zero submagma* if  $f^{-1}(0) = 0$  and for all  $(g, h)$  and  $(g', h')$  in  $f$  with  $gg' \neq 0$ , the element  $(gg', hh')$  belongs to  $f$ . Let  $S_0(G \times H)$  denote the partially ordered set of zero submagmas of  $G \times H$  ordered by inclusion. A function  $f : G \rightarrow H$  is called a *zero magma homomorphism* if the graph of  $f$  considered as a subset of  $G \times H$  belongs to  $S_0(G \times H)$ . The collection of zero magma homomorphisms  $G \rightarrow H$  is denoted  $\text{hom}_0(G, H)$ . Note that a function  $f : G \rightarrow H$  with the property that  $f^{-1}(0) = 0$  is a zero magma homomorphism precisely when  $f(gg') = f(g)f(g')$  for all  $g, g' \in G$  with  $gg' \neq 0$ . We let  $G_0(R)$  denote the partially ordered set of  $G$ -filters  $(W_g)_{g \in G}$  on  $R$  satisfying  $W_0 = V_0$ . We say that such a  $G$ -filter is *nonzero* if  $W_g \neq \{0\}$  for all nonzero  $g \in G$ . A grading  $(W_g)_{g \in G}$  on  $R$  is called *nonzero* if it is nonzero as a  $G$ -filter on  $R$ .

**Remark 1.** Not all zero magma homomorphisms are magma homomorphisms. In fact, let  $G = \{a, b, 0\}$  and  $H = \{c, 0\}$  be two magmas equipped with rules of composition defined by  $aa = a$ ,  $bb = b$ ,  $ab = ba = a0 = 0a = b0 = 0b = 00 = 0$  and  $cc = c$ ,  $c0 = 0c = 00 = 0$  respectively. Define a function  $f : G \rightarrow H$  by  $f(a) = f(b) = c$  and  $f(0) = 0$ . Since  $f(aa) = f(a) = c = cc = f(a)f(a)$  and  $f(bb) = f(b) = c = cc = f(b)f(b)$  it follows that  $f$  is a zero magma homomorphism. However, since  $f(ab) = f(0) = 0 \neq c = cc = f(a)f(b)$ ,  $f$  is not a magma homomorphism.

**Proof of Theorem 3 and Theorem 4.** By Proposition 3 and Proposition 4 we get that  $MF = \text{id}_{S(G \times H)}$  and  $FM = \text{id}_{H_0(R)}$  respectively. Therefore the "nonzero" version of Theorem 4 follows. The "nonzero" version of Theorem 3 follows from the same propositions by restriction of the maps  $F$  and  $M$  to respectively  $\text{hom}(G, H)$  and the set of elementary  $H$ -gradings on  $R$ . The "zero" versions of Theorem 4 and Theorem 3 follow from the easily checked fact that  $M_0F_0 = \text{id}_{S_0(G \times H)}$  and  $F_0M_0 = \text{id}_{H_0(R)}$  where  $F_0$  and  $M_0$  denote the restrictions of  $F$  and  $M$  to  $S_0(G \times H)$  and  $H_0(R)$  respectively.  $\square$

**Definition 4.** Let  $G$  be a magma and  $K$  a field. Recall that the *magma algebra*  $K[G]$  of  $G$  over  $K$  is the collection of formal sums  $\sum_{g \in G} a_g g$ , where  $a_g \in K$ , for  $g \in G$ , are chosen so that all but finitely many of them are nonzero. The addition and multiplication on  $K[G]$  is defined by  $\sum_{g \in G} a_g s + \sum_{t \in G} b_t t = \sum_{g \in G} (a_g + b_g)g$  and  $\sum_{s \in G} a_s s \sum_{t \in G} b_t t = \sum_{g \in G} \sum_{st=g} a_s b_t g$  respectively, for all  $\sum_{s \in G} a_s s \in K[G]$  and all  $\sum_{t \in G} b_t t \in K[G]$ . For the rest of the article, we fix the strong  $G$ -grading  $(V_g)_{g \in G}$  on  $K[G]$  defined by putting  $V_g = Kg$  for all  $g \in G$ .

**Corollary 1.** *If  $G$  and  $H$  are (zero) magmas, then there is a bijection between the set of elementary (zero)  $H$ -gradings on  $K[G]$  and the set of (zero) magma homomorphisms from  $G$  to  $H$ .*

*Proof.* This follows immediately from Theorem 3.  $\square$



**Corollary 2.** *If  $G$  and  $H$  are (zero) magmas, then there is an isomorphism between the preordered set of elementary (zero)  $H$ -filters on  $K[G]$  and the preordered set of (zero) submagmas of  $G \times H$ .*

*Proof.* This follows immediately from Theorem 4.  $\square$

**Example 1.** Suppose that  $G$  and  $H$  are finite groups. By Corollary 1, the cardinality of the set of elementary  $H$ -gradings on  $K[G]$  equals  $|\text{hom}(G, H)|$ . The latter seems hard to compute for general groups (see e.g. [1] and [24] for some results concerning congruences involving this number). However, for abelian groups  $G$  and  $H$  it is fairly easy. In fact, if  $G$  and  $H$  are cyclic, then  $|\text{hom}(G, H)|$  equals the greatest common divisor of  $|G|$  and  $|H|$ . For the case of general abelian  $G$  and  $H$ , suppose that  $p_i$  denotes the  $i$ th prime number and  $G = \bigoplus_{i \geq 1} \bigoplus_{j \geq 1} \left(\mathbb{Z}_{p_i^j}\right)^{a_{ij}}$  and  $H = \bigoplus_{k \geq 1} \bigoplus_{l \geq 1} \left(\mathbb{Z}_{p_k^l}\right)^{a_{kl}}$ , where  $a_{ij} = a_{kl} = 0$  for all but finitely many  $i, j, k$  and  $l$ . Then  $|\text{hom}(G, H)| = \prod_{i \geq 1} \prod_{j, k \geq 1} \left(p_i^{\min(j, k)}\right)^{a_{ij} a_{ik}}$ .

**Remark 2.** From the discussion in Example 1 it follows that  $|\text{hom}(G, H)| = |\text{hom}(H, G)|$  for all finite abelian groups  $G$  and  $H$ . It is not clear to the author at present whether this equality is true for all groups  $G$  and  $H$ .

**Example 2.** Suppose that  $G$  and  $H$  are finite groups. It is easy to see that a submagma of a finite group is a subgroup. Therefore, by Corollary 2, the number of elementary  $H$ -filters on  $K[G]$  equals the number of subgroups of  $G \times H$ . The latter seems hard to determine (see [23] for a discussion on the structure of the lattice of subgroups of  $G \times H$ ). Therefore we here only discuss two particular cases of interest.

If  $|G|$  and  $|H|$  are relatively prime, then it is easy to see that the subgroup lattice of  $G \times H$  equals the product of the subgroup lattices of  $G$  and  $H$ . In particular it follows that the number of subgroups of  $G \times H$  equals the product of the number of subgroups of  $G$  and the number of subgroups of  $H$ .

If  $G$  and  $H$  both are direct products of copies of  $\mathbb{Z}_p$ , for some prime number  $p$ , then  $G \times H$  is a vector space over  $\mathbb{Z}_p$ . By a simple counting argument, the number of  $\mathbb{Z}_p$ -subspaces of  $\mathbb{Z}_p^n$  is  $\sum_{k=1}^n A_k / B_k$  where  $A_k = (p^n - 1)(p^n - p) \dots (p^n - p^{k-1})$  and  $B_k = (p^k - 1)(p^k - p) \dots (p^k - p^{k-1})$ .

**Example 3.** There are 10 nonisomorphic magmas of order two. In fact, let the two elements of such a magma be denoted  $a$  and  $b$  and let the notation  $x_1 x_2 x_3 x_4$  mean that the magma operation is defined by the relations  $a^2 = x_1$ ,  $ab = x_2$ ,  $ba = x_3$  and  $b^2 = x_4$ . Furthermore, let  $t$  denote the nonidentity bijection from  $\{a, b\}$  to  $\{a, b\}$ . Then it is easy to see that  $t : x_1 x_2 x_3 x_4 \rightarrow y_1 y_2 y_3 y_4$  precisely when  $y_1 = t(x_4)$ ,  $y_2 = t(x_3)$ ,  $y_3 = t(x_2)$  and  $y_4 = t(x_1)$ . Using this it is a straightforward task to verify that a complete set of representatives for the different isomorphism classes of magmas of order two is  $aaaa$ ,  $baaa$ ,  $abaa$ ,  $aaba$ ,  $aaab$ ,  $aabb$ ,  $bbaa$ ,  $abab$ ,  $baba$  and  $abba$ . Now we determine  $\text{hom}(G, H)$  for all 100 choices of two element magmas  $G$  and  $H$ . In this description, we let  $x_1 x_2 x_3 x_4 y_1 y_2 y_3 y_4$  denote the set

$\text{hom}(x_1x_2x_3x_4, y_1y_2y_3y_4)$ . We let 1 denote the identity function  $\{a, b\} \rightarrow \{a, b\}$  and we let  $a$  (or  $b$ ) denote the constant function  $\{a, b\} \rightarrow \{a, b\}$  that maps both  $a$  and  $b$  to  $a$  (or  $b$ ).

$$\begin{aligned}
aaaaaaaa &= \{1, a\} & aaaabaaa &= \emptyset & aaaaabaa &= \{a\} & aaaaaaba &= \{a\} \\
aaaaaaab &= \{a\} & aaaaaabb &= \{a, b\} & aaaabbaa &= \emptyset & aaaaabab &= \{a, b\} \\
& & aaaababa &= \emptyset & aaaaabba &= \{a\} & & \\
baaaaaaa &= \{a\} & baaaabaa &= \{1\} & baaaabaa &= \{a\} & baaaaaba &= \{a\} \\
baaaaaab &= \{a, b\} & baaaaabb &= \{a, b\} & baaabbaa &= \emptyset & baaaabab &= \{a, b\} \\
& & baaababa &= \emptyset & baaaabba &= \{a\} & & \\
abaaaaaa &= \{a\} & abaabaaa &= \emptyset & abaaabaa &= \{1, a\} & abaaaaba &= \{a\} \\
abaaaaab &= \{a, b\} & abaaaabb &= \{a, b\} & abaabbaa &= \emptyset & abaaabab &= \{a, b\} \\
& & abaababa &= \emptyset & abaaabba &= \{a\} & & \\
aabaaaaa &= \{a\} & aababaaa &= \emptyset & aabaabaa &= \{a\} & aabaaaba &= \{1, a\} \\
aabaaaab &= \{a, b\} & aabaaabb &= \{a, b\} & aababbaa &= \emptyset & aabaabab &= \{a, b\} \\
& & aabababa &= \emptyset & aabaabba &= \{a\} & & \\
aaabaaaa &= \{a\} & aaabbaaa &= \emptyset & aaababaa &= \{a\} & aaabaaba &= \{a\} \\
aaabaaab &= \{1, a, b\} & aaabaabb &= \{a, b\} & aaabbbba &= \emptyset & aaababab &= \{a, b\} \\
& & aaabbaba &= \emptyset & aaababba &= \{a\} & & \\
aabbaaaa &= \{a\} & aabbbaaa &= \emptyset & aabbabaa &= \{a\} & aabbbaaba &= \{a\} \\
aabbaaab &= \{a, b\} & aabbbaabb &= \{1, a, b, t\} & aabbbbaa &= \emptyset & aabbabab &= \{a, b\} \\
& & aabbbaba &= \emptyset & aabbabba &= \{a\} & & \\
bbaaaaaa &= \{a\} & bbaabaaa &= \emptyset & bbaaabaa &= \{a\} & bbaaaaba &= \{a\} \\
bbaaaaab &= \{a, b\} & bbaaaabb &= \{a, b\} & bbaabbaa &= \{1, t\} & bbaaabab &= \{a, b\} \\
& & bbaababa &= \emptyset & bbaaabba &= \{a\} & & \\
ababaaaa &= \{a\} & ababbaaa &= \emptyset & abababaa &= \{a\} & ababaaba &= \{a\} \\
ababaaab &= \{a, b\} & ababaabb &= \{a, b\} & ababbbba &= \emptyset & abababab &= \{1, a, b, t\} \\
& & baba &= \emptyset & abababba &= \{a\} & & \\
babaaaaa &= \{a\} & bababaaa &= \emptyset & babaabaa &= \{a\} & babaaaba &= \{a\} \\
babaaaab &= \{a, b\} & babaaabb &= \{a, b\} & bababbaa &= \emptyset & babaabab &= \{a, b\} \\
& & babababa &= \{1, t\} & babaabba &= \{a\} & & \\
abbaaaaa &= \{a\} & abbabaaa &= \emptyset & abbaabaa &= \{a\} & abbbaaba &= \{a\} \\
abbaaaab &= \{a, b\} & abbaaabb &= \{a, b\} & abbabbaa &= \emptyset & abbaabab &= \{a, b\} \\
& & abbababa &= \emptyset & abbaabba &= \{1, a\} & &
\end{aligned}$$

By Theorem 3 there is a bijection between the set of elementary  $H$ -gradings on  $K[G]$  and  $\text{hom}(G, H)$ . This result can be applied on the list of hom-sets above. For instance the following results hold.

- There are no elementary  $baaa$ -gradings on  $K[aaaa]$ .
- There is precisely one elementary  $abaa$ -grading on  $K[aaaa]$ , namely the constant grading  $a$ .
- There are no nonidentity elementary  $baaa$ -gradings on  $K[baaa]$ .
- There are precisely two elementary  $aabb$ -gradings on  $K[baaa]$ , namely the constant gradings  $a$  and  $b$ .



- There are no two element magmas  $G$  and  $H$  such that there are precisely three different elementary  $H$ -gradings on  $K[G]$ .
- There are precisely four different elementary  $aabb$ -gradings on  $K[aabb]$ , namely the identity grading 1, the constant gradings  $a$  and  $b$  and the grading defined by the twist map  $t$ .

**Example 4.** Let  $G$  and  $H$  be magmas of order two. We use the notation from Example 3. By Theorem 4, there is an isomorphism between the preordered set of elementary  $H$ -filters on  $K[G]$  and the preordered set of submagmas of  $G \times H$ . We apply this isomorphism in some particular cases. Recall that if  $f$  is a submagma of  $G \times H$ , then we let  $F(f)$  denote the corresponding  $H$ -filter of  $K[G]$ .

Suppose that  $G = H = aaaa$ . Then the preordered set of submagmas of  $G \times H$  consists of the empty set and all the eight subsets of  $G \times H$  containing  $(a, a)$ . These correspond to nine different elementary  $H$ -filters on  $K[G]$ , namely:

$$\begin{aligned}
F(\emptyset)_a &= \{0\} & F(\emptyset)_b &= \{0\} \\
F(\{(a, a)\})_a &= Ka & F(\{(a, a)\})_b &= \{0\} \\
F(\{(a, a), (a, b)\})_a &= Ka & F(\{(a, a), (a, b)\})_b &= Ka \\
F(\{(a, a), (b, a)\})_a &= Ka + Kb & F(\{(a, a), (b, a)\})_b &= \{0\} \\
F(\{(a, a), (b, b)\})_a &= Ka & F(\{(a, a), (b, b)\})_b &= Kb \\
F(\{(a, a), (a, b), (b, a)\})_a &= Ka + Kb & F(\{(a, a), (a, b), (b, a)\})_b &= Ka \\
F(\{(a, a), (a, b), (b, b)\})_a &= Ka & F(\{(a, a), (a, b), (b, b)\})_b &= Ka + Kb \\
F(\{(a, a), (b, a), (b, b)\})_a &= Ka + Kb & F(\{(a, a), (b, a), (b, b)\})_b &= Kb \\
F(\{(a, a), (a, b), (b, a), (b, b)\})_a &= Ka + Kb & F(\{(a, a), (a, b), (b, a), (b, b)\})_b &= Ka + Kb
\end{aligned}$$

Suppose that  $G = H = abba$ . Then the preordered set of submagmas of  $G \times H$  consists of the empty set and all the five subgroups of  $G \times H$ . These correspond to six different elementary  $H$ -filters on  $K[G]$ , namely:

$$\begin{aligned}
F(\emptyset)_a &= \{0\} & F(\emptyset)_b &= \{0\} \\
F(\{(a, a)\})_a &= Ka & F(\{(a, a)\})_b &= \{0\} \\
F(\{(a, a), (a, b)\})_a &= Ka & F(\{(a, a), (a, b)\})_b &= Ka \\
F(\{(a, a), (b, a)\})_a &= Ka + Kb & F(\{(a, a), (b, a)\})_b &= \{0\} \\
F(\{(a, a), (b, b)\})_a &= Ka & F(\{(a, a), (b, b)\})_b &= Kb \\
F(\{(a, a), (a, b), (b, a), (b, b)\})_a &= Ka + Kb & F(\{(a, a), (a, b), (b, a), (b, b)\})_b &= Ka + Kb
\end{aligned}$$

Suppose that  $G = abaa$  and  $H = aabb$ . Then there are six submagmas of  $G \times H$  which in turn correspond to the following  $H$ -filters of  $K[G]$ :

$$\begin{aligned}
F(\emptyset)_a &= \{0\} & F(\emptyset)_b &= \{0\} \\
F(\{(a, a)\})_a &= Ka & F(\{(a, a)\})_b &= \{0\} \\
F(\{(a, b)\})_a &= \{0\} & F(\{(a, b)\})_b &= Ka \\
F(\{(a, a), (a, b)\})_a &= Ka & F(\{(a, a), (a, b)\})_b &= Ka \\
F(\{(a, a), (b, a)\})_a &= Ka + Kb & F(\{(a, a), (b, a)\})_b &= \{0\} \\
F(\{(a, a), (a, b), (b, a), (b, b)\})_a &= Ka + Kb & F(\{(a, a), (a, b), (b, a), (b, b)\})_b &= Ka + Kb
\end{aligned}$$

**Example 5.** Suppose that  $n$  is a positive integer and let  $G_n$  denote the zero magma having 0 and the matrices  $e_{i,j}$  (see Section 1), for  $1 \leq i, j \leq n$ , as elements. Let  $m$  be another positive integer. Now we describe all nonzero elementary  $G_n$ -gradings of  $K[G_m]$ . Take  $f \in \text{hom}_0(G_m, G_n)$ . It is easy to see that  $f$  is defined by  $f(0) = 0$  and  $f(e_{i,j}) = e_{p(i),p(j)}$ , for  $1 \leq i, j \leq m$ , for a unique function  $p : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ . By Theorem 3  $f$  corresponds to the nonzero  $G_n$ -grading of  $K[G_m]$  defined by

$$K[G_m]_{e_{i,j}} = \sum_{\substack{k \in p^{-1}(i) \\ l \in p^{-1}(j)}} K e_{k,l}$$

for  $1 \leq i, j \leq n$ . In particular, there are  $n^m$  different nonzero  $G_n$ -gradings of  $G_m$ .

**Example 6.** Suppose that  $n$  is a positive integer and let  $G_n$  be the zero magma from Example 5. Let  $H$  be a zero magma with the property that its nonzero elements form a group with  $q$  elements. Now we describe all nonzero elementary  $H$ -gradings of  $K[G_n]$ . Take  $f \in \text{hom}_0(G_n, H)$ . Since  $G_n$  is generated, as a zero magma, by  $\{e_{i,i+1} \mid i = 1, \dots, n-1\}$  it is clear that  $f$  is determined by the  $q-1$  elements  $h_i = f(e_{i,i+1})$ , for  $i = 1, \dots, i-1$ . By Theorem 3  $f$  corresponds to the nonzero  $H$ -grading of  $K[G_n]$  defined by  $K[G_n]_h = \sum_{e_{i,j} \in f^{-1}(h)} K e_{i,j}$ , for all  $h \in H$ . In particular, there are  $q^{n-1}$  different nonzero elementary  $H$ -gradings on  $K[G_n]$ .

### 3. CATEGORY FILTERED RINGS

In this section, we use the results from Section 2 to prove category versions of Theorem 3 and Theorem 4 (see Theorem 5 and Theorem 6). Then we apply these results to some particular cases of category algebras (see Examples 7-11). In the end of this section, we prove Theorem 2.

**Definition 5.** We define a *precategory* exactly as a category except that it does not necessarily have an identity element at each object. We define a *subprecategory* of a precategory exactly as a subcategory except that possible identity elements at objects of the subprecategory need not necessarily belong to the morphisms of the subprecategory. Recall from Freyd [9] that a *prefunctor*, between (pre)categories, is defined exactly as a functor except that it does not necessarily respect identity elements.

**Definition 6.** Now we recall some old and define some new notions from the theory of category graded rings (also see e.g. [14], [15], [16], [19], [20] or [21]). Let  $R$  be a ring and  $\Gamma$  a category. A  $\Gamma$ -filter on  $R$  is a set of additive subgroups,  $(V_s)_{s \in \text{mor}(\Gamma)}$  of  $R$  such that for all  $s, t \in \text{mor}(\Gamma)$ , we have  $V_s V_t \subseteq V_{st}$ , if  $(s, t) \in \Gamma^{(2)}$ , and  $V_s V_t = \{0\}$  otherwise. In this article, we say that such a filter is nonzero (or strong) if  $V_s \neq \{0\}$  (or  $V_s V_t = V_{st}$ ) for all  $s \in G$  (or all  $(s, t) \in \Gamma^{(2)}$ ). The ring  $R$  is called  $\Gamma$ -graded if there is a  $\Gamma$ -filter,  $(V_s)_{s \in \text{mor}(\Gamma)}$  on  $R$  such that  $R = \bigoplus_{s \in \text{mor}(\Gamma)} V_s$ . If  $R$  is  $\Gamma$ -graded and nonzero

(strong) as a  $\Gamma$ -system, then it is called nonzero (strongly)  $\Gamma$ -graded. If  $\Lambda$  is another category, then we say that a  $\Lambda$ -filter  $(W_t)_{t \in \text{mor}(\Lambda)}$  on  $R$  is *elementary* if  $W_t = \sum_{V_s \subseteq W_t} V_s$  for all  $t \in \text{mor}(\Lambda)$ .

**Theorem 5.** *Let  $\Gamma$  and  $\Lambda$  be categories and  $R$  a ring equipped with a nonzero strong  $\Gamma$ -grading. (a) There is a bijection between the set of elementary  $\Lambda$ -gradings on  $R$  and the set of prefunctors from  $\Gamma$  to  $\Lambda$ . (b) If  $\Lambda$  has the property that for each  $e \in \text{ob}(\Gamma)$ , the only idempotent of  $\Gamma_e$  is the identity element from  $e$  to  $e$ , then there is a bijection between the set of elementary  $\Lambda$ -gradings on  $R$  and the set of functors from  $\Gamma$  to  $\Lambda$ . The last conclusion holds in particular if  $\Lambda$  is a groupoid.*

*Proof.* Suppose that  $R$  is equipped with the nonzero and strong  $\Gamma$ -grading  $(V_s)_{s \in \text{mor}(\Gamma)}$ . Let  $G$  be the magma having  $\text{mor}(\Gamma) \cup \{0\}$  as elements. If we put  $V_0 = \{0\}$ , then  $(V_g)_{g \in G}$  is a nonzero magma grading on  $R$ .

(a) Let  $(W_t)_{t \in \text{mor}(\Lambda)}$  be an elementary  $\Lambda$ -grading on  $R$ . If we put  $H = \text{mor}(\Lambda) \cup \{0\}$  and  $W_0 = \{0\}$ , then  $(W_h)_{h \in H}$  is a nonzero elementary magma grading on  $R$ . By Theorem 3 this  $H$ -grading uniquely defines a zero magma homomorphism  $M((W_h)_{h \in H})$  from  $G$  to  $H$ . But a such a homomorphism restricts uniquely to a prefunctor from  $\Gamma$  to  $\Lambda$ .

On the other hand, given a prefunctor from  $\Gamma$  to  $\Lambda$ , then it extends uniquely to a zero magma homomorphism from  $G$  to  $H$ . This magma homomorphism defines, by Theorem 3 again, a unique nonzero elementary magma grading  $(W_h)_{h \in H}$  on  $R$ .

It is clear that these constructions are inverse to each other and hence defines a bijection between the set of nonzero elementary  $\Lambda$ -gradings on  $R$  and the set of prefunctors from  $\Gamma$  to  $\Lambda$ .

(b) The first part of the statement follows immediately from (a). The second part of the statement follows from the first part and the fact that the prefunctor image of an identity element is an idempotent and therefore, by the assumptions, an identity element. The last part of the statement follows from the second part and the fact that the only idempotents in groups are the identity elements.  $\square$

**Theorem 6.** *If  $\Gamma$  and  $\Lambda$  are categories and  $R$  is a ring equipped with a nonzero strong  $\Gamma$ -grading, then there is an isomorphism between the preordered set of elementary  $\Lambda$ -filters on  $R$  and the preordered set of subprecategories of  $\Gamma \times \Lambda$ .*

*Proof.* Suppose that  $R$  is equipped with the nonzero and strong  $\Gamma$ -grading  $(V_s)_{s \in \text{mor}(\Gamma)}$ . Let  $G$  be the magma having  $\text{mor}(\Gamma) \cup \{0\}$  as elements. If we put  $V_0 = \{0\}$ , then  $(V_g)_{g \in G}$  is a nonzero magma grading on  $R$ .

Let  $(W_t)_{t \in \text{mor}(\Lambda)}$  be a nonzero elementary  $\Lambda$ -filter on  $R$ . If we put  $H = \text{mor}(\Lambda) \cup \{0\}$  and  $W_0 = \{0\}$ , then  $(W_h)_{h \in H}$  is a nonzero elementary magma filter on  $R$ . By Theorem 4 this  $H$ -filter uniquely defines a zero submagma  $M((W_h)_{h \in H})$  of  $G \times H$ . But the nonzero elements of a zero submagma of  $G \times H$  uniquely defines a subprecategory of  $\Gamma \times \Lambda$ .

On the other hand, given a subprecategory of  $\Gamma \times \Lambda$ , then it extends uniquely to a zero submagma of  $G \times H$ . This zero submagma defines, by Theorem 4 again, a unique nonzero elementary magma filter  $(W_h)_{h \in H}$  on  $R$ .

It is clear that these constructions are inverse to each other and that they respect inclusion. Therefore it defines an isomorphism between the preordered set of  $\Lambda$ -filters on  $R$  and the preordered set of subprecategories of  $\Gamma \times \Lambda$ .  $\square$

**Example 7.** Suppose that  $\Gamma$  and  $\Lambda$  are finite, connected and thin categories. If  $m = |\text{ob}(\Gamma)|$  and  $n = |\text{ob}(\Lambda)|$ , then there are  $n^m$  elementary  $\Lambda$ -gradings on  $K[\Gamma]$ .

Indeed, by Theorem 5, there is a bijection between the set of elementary  $\Lambda$ -gradings on  $K[\Gamma]$  and the set of functors  $\Gamma \rightarrow \Lambda$ . But since  $\Gamma$  and  $\Lambda$  are connected and thin, such a functor is completely determined by its restriction to  $\text{ob}(\Gamma)$ . So we seek the cardinality of the set of functions from  $\{1, \dots, m\}$  to  $\{1, \dots, n\}$ . Therefore there are exactly  $n^m$  different functors  $\Gamma \rightarrow \Lambda$ .

Of these gradings, precisely  $\frac{1}{n!} \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^m$  are nonzero. In fact, this follows from the above and the observation that an elementary  $\Lambda$ -grading on  $K[\Gamma]$  is nonzero precisely when the corresponding functor  $\Gamma \rightarrow \Lambda$  is surjective on objects. So we seek the cardinality of the set of surjective functions from  $\{1, \dots, m\}$  to  $\{1, \dots, n\}$ . But this is the so called Stirling number which can be calculated in accordance with the claim (see e.g. [22]).

**Remark 3.** The first part of Example 7 also follows directly from Example 5 by noting that if we adjoin a zero element to the unique connected thin category with  $n$  elements, then we get the magma  $G_n$ .

**Remark 4.** It is easy to see that if  $\Gamma$  and  $\Lambda$  are finite, connected and thin categories, then an elementary  $\Lambda$ -grading of  $K[\Gamma]$  is nonzero if and only if it is strong. Therefore, the second part of Example 7 also counts the number of strong elementary  $\Lambda$ -gradings on  $K[\Gamma]$ .

**Example 8.** Let  $\Gamma$  be the category having two elements  $a$  and  $b$  and non-identity morphisms  $\alpha : a \rightarrow a$ ,  $\beta : a \rightarrow b$  and  $\gamma : a \rightarrow b$  subject to the following relations  $\alpha^2 = \text{id}_a$  and  $\beta\alpha = \gamma$ . Let  $\Lambda$  be the category having one object  $c$  and one nonidentity morphism  $\delta : c \rightarrow c$  subject to the relation  $\delta^2 = \text{id}_c$ . Note that this means that  $\Lambda$  is the group with two elements. There are four different functors  $F : \Gamma \rightarrow \Lambda$ . Indeed,  $F$  must map all objects of  $\Lambda$  to  $c$ . There are four different ways for  $F$  to map the morphisms of  $\Gamma$ , namely:

$$\begin{aligned} F(\alpha) &= F(\beta) = F(\gamma) = \text{id}_c \\ F(\alpha) &= \delta \quad F(\beta) = \text{id}_c \quad F(\gamma) = \delta \\ F(\alpha) &= \text{id}_c \quad F(\beta) = \delta \quad F(\gamma) = \delta \\ F(\alpha) &= \delta \quad F(\beta) = \delta \quad F(\gamma) = \text{id}_c. \end{aligned}$$

These four functors correspond, by Theorem 5(b), to the following elementary  $\Lambda$ -gradings of  $K[\Gamma]$ :

$$\begin{aligned} K[\Gamma]_{\text{id}_c} &= K[\Gamma] & K[\Gamma]_\delta &= \{0\} \\ K[\Gamma]_{\text{id}_c} &= K\text{id}_a + K\text{id}_b + K\beta & K[\Gamma]_\delta &= K\alpha + K\gamma \\ K[\Gamma]_{\text{id}_c} &= K\text{id}_a + K\text{id}_b + K\alpha & K[\Gamma]_\delta &= K\beta + K\gamma \\ K[\Gamma]_{\text{id}_c} &= K\text{id}_a + K\text{id}_b + K\gamma & K[\Gamma]_\delta &= K\alpha + K\beta \end{aligned}$$

**Example 9.** Let the categories  $\Gamma$  and  $\Lambda$  be defined as in Example 8. Now we illustrate Theorem 6 by considering two subprecategories of  $\Gamma \times \Lambda$ , both of them having  $\text{ob}(\Gamma) \times \text{ob}(\Lambda)$  as set of objects.

First suppose that the set of morphisms of the subprecategory is:

$$\{(\text{id}_a, \text{id}_c), (\beta, \text{id}_c), (\beta, \delta), (\text{id}_b, \text{id}_c), (\text{id}_b, \delta)\}$$

This corresponds to the following elementary  $\Lambda$ -filter on  $K[\Gamma]$ :

$$W_{\text{id}_c} = K\text{id}_a + K\text{id}_b + K\beta \quad W_\delta = K\text{id}_b + K\beta$$

Next suppose that the set of morphisms of the subprecategory is:

$$\{(\text{id}_a, \text{id}_c), (\text{id}_a, \delta), (\alpha, \text{id}_c), (\alpha, \delta), (\beta, \text{id}_c), (\beta, \delta), (\gamma, \text{id}_c), (\gamma, \delta), (\text{id}_b, \text{id}_c)\}$$

This corresponds to the following elementary  $\Lambda$ -filter on  $K[\Gamma]$ :

$$W_{\text{id}_c} = K\text{id}_a + K\text{id}_b + K\alpha + K\beta + K\gamma \quad W_\delta = K\text{id}_a + K\text{id}_b + K\alpha + K\beta + K\gamma$$

**Example 10.** Let the category  $\Gamma$  be defined as in Example 8. Let  $\Lambda$  be the category having one object  $c$  and one nonidentity morphism  $\delta : c \rightarrow c$  subject to the relation  $\delta^2 = \delta$ . There are eight different prefunctors  $F : \Gamma \rightarrow \Lambda$ . Indeed,  $F$  must map all objects of  $\Lambda$  to  $c$ . There are eight different ways for  $F$  to map the morphisms of  $\Gamma$ , namely:

$$\begin{aligned} F(\alpha) &= \text{id}_c & F(\beta) &= \text{id}_c & F(\text{id}_b) &= \text{id}_c & F(\gamma) &= \text{id}_c & F(\text{id}_a) &= \text{id}_c \\ F(\alpha) &= \delta & F(\beta) &= \text{id}_c & F(\text{id}_b) &= \text{id}_c & F(\gamma) &= \delta & F(\text{id}_a) &= \delta \\ F(\alpha) &= \text{id}_c & F(\beta) &= \delta & F(\text{id}_b) &= \text{id}_c & F(\gamma) &= \delta & F(\text{id}_a) &= \text{id}_c \\ F(\alpha) &= \text{id}_c & F(\beta) &= \text{id}_c & F(\text{id}_b) &= \delta & F(\gamma) &= \text{id}_c & F(\text{id}_a) &= \text{id}_c \\ F(\alpha) &= \delta & F(\beta) &= \delta & F(\text{id}_b) &= \text{id}_c & F(\gamma) &= \delta & F(\text{id}_a) &= \delta \\ F(\alpha) &= \delta & F(\beta) &= \text{id}_c & F(\text{id}_b) &= \delta & F(\gamma) &= \delta & F(\text{id}_a) &= \delta \\ F(\alpha) &= \text{id}_c & F(\beta) &= \delta & F(\text{id}_b) &= \delta & F(\gamma) &= \delta & F(\text{id}_a) &= \text{id}_c \\ F(\alpha) &= \delta & F(\beta) &= \delta & F(\text{id}_b) &= \delta & F(\gamma) &= \delta & F(\text{id}_a) &= \delta \end{aligned}$$

These eight functors correspond, by Theorem 5(a), to the following elementary  $\Lambda$ -gradings of  $K[\Gamma]$ :

$$\begin{aligned} K[\Gamma]_{\text{id}_c} &= K[\Gamma] & K[\Gamma]_\delta &= \{0\} \\ K[\Gamma]_{\text{id}_c} &= K\text{id}_b + K\beta & K[\Gamma]_\delta &= K\alpha + K\gamma + K\text{id}_a \\ K[\Gamma]_{\text{id}_c} &= K\text{id}_a + K\text{id}_b + K\alpha & K[\Gamma]_\delta &= K\beta + K\gamma \\ K[\Gamma]_{\text{id}_c} &= K\text{id}_a + K\alpha + K\beta + K\gamma & K[\Gamma]_\delta &= K\text{id}_b \\ K[\Gamma]_{\text{id}_c} &= K\text{id}_b & K[\Gamma]_\delta &= K\alpha + K\beta + K\gamma + K\text{id}_a \end{aligned}$$

$$\begin{aligned}
K[\Gamma]_{\text{id}_c} &= K\beta & K[\Gamma]_\delta &= K\alpha + K\gamma + K\text{id}_a + K\text{id}_b \\
K[\Gamma]_{\text{id}_c} &= K\text{id}_a + K\alpha & K[\Gamma]_\delta &= K\beta + K\gamma + K\text{id}_b \\
K[\Gamma]_{\text{id}_c} &= \{0\} & K[\Gamma]_\delta &= K[\Gamma]
\end{aligned}$$

**Example 11.** Let the categories  $\Gamma$  and  $\Lambda$  be defined as in Example 10. Now we again illustrate Theorem 6 by considering two subprecategories of  $\Gamma \times \Lambda$ , both of them having  $\text{ob}(\Gamma) \times \text{ob}(\Lambda)$  as set of objects.

First suppose that the set of morphisms of the subprecategory is:

$$\{(\text{id}_a, \text{id}_c), (\beta, \text{id}_c), (\beta, \delta), (\text{id}_b, \delta)\}$$

This corresponds to the following elementary  $\Lambda$ -filter on  $K[\Gamma]$ :

$$W_{\text{id}_c} = K\text{id}_a + K\beta \quad W_\delta = K\text{id}_b + K\beta$$

Next suppose that the set of morphisms of the subprecategory is:

$$\{(\text{id}_a, \text{id}_c), (\text{id}_a, \delta), (\alpha, \text{id}_c), (\alpha, \delta), (\beta, \delta), (\gamma, \delta), (\text{id}_b, \text{id}_c)\}$$

This corresponds to the following elementary  $\Lambda$ -filter on  $K[\Gamma]$ :

$$W_{\text{id}_c} = K\text{id}_a + K\text{id}_b + K\alpha \quad W_\delta = K\text{id}_a + K\text{id}_b + K\alpha + K\beta + K\gamma$$

**Proof of Theorem 2.** Suppose that  $\Gamma$  and  $\Lambda$  are finite connected groupoids. By Theorem 5(b), there is a bijection between the set of elementary  $\Lambda$ -gradings on  $K[\Gamma]$  and the set of functors from  $\Gamma$  to  $\Lambda$ . Therefore we now set out to determine the cardinality of the latter set.

First note that  $\Gamma_e \cong \Gamma_{e'}$  as groups for all  $e, e' \in \text{ob}(\Gamma)$ . In fact, since  $\Gamma$  is connected, there is  $E : e \rightarrow e'$  in  $\text{mor}(\Gamma)$ . Now it is easy to see that the map  $\Gamma_e \ni x \mapsto ExE^{-1} \ni \Gamma_{e'}$  is an isomorphism of groups. Therefore, the cardinality of  $\text{hom}(\Gamma_e, \Gamma_{e'})$  is the same for all choices of  $e, e' \in \text{ob}(\Gamma)$ , and the cardinality of  $\text{hom}(\Gamma_e, \Lambda_f)$  is the same for all choices of  $e \in \text{ob}(\Gamma)$  and  $f \in \text{ob}(\Lambda)$ .

Next, fix  $i \in \text{ob}(\Gamma)$ . For each  $j \in \text{ob}(\Gamma)$ , with  $j \neq i$ , fix a morphism  $\alpha_{ij} : j \rightarrow i$ . Then  $\Gamma$  is generated, as a groupoid, by  $X = \Gamma_i \cup \{\alpha_{ij} \mid j \in \text{ob}(\Gamma), j \neq i\}$ . In fact, suppose that  $\beta : k \rightarrow l$  is a morphism in  $\Gamma$ . Then  $\alpha_{il}\beta\alpha_{ik}^{-1} \in \Gamma_i$  and hence  $\beta \in \alpha_{il}^{-1}\Gamma_i\alpha_{ik}$ . It is also clear that there are no nontrivial products of elements (and their inverses) in  $X$  that yields an identity element. Therefore a functor  $\Gamma \rightarrow \Lambda$  is completely determined by its action on  $X$ . Suppose that  $e \in \text{ob}(\Gamma)$  and  $f \in \text{ob}(\Lambda)$  and put  $m = |\text{ob}(\Lambda)|$ ,  $n = |\text{ob}(\Gamma)|$ ,  $p = |\text{hom}(\Gamma_e, \Lambda_f)|$  and  $q = |\text{hom}(\Gamma_e, \Gamma_e)|$ . There are  $m^n$  ways for the functor to map identity elements. For each such choice there are  $pq^{n-1}$  ways to map the elements of  $X$ . Therefore, there are  $(pq^{n-1})^{m^n}$  elementary  $\Lambda$ -gradings on the groupoid algebra  $K[\Gamma]$ .  $\square$

**Remark 5.** If  $\Gamma$  and  $\Lambda$  are arbitrary (not necessarily connected) finite groupoids, then  $\Gamma = \biguplus_{i \in I} \Gamma_i$  and  $\Lambda = \biguplus_{j \in J} \Lambda_j$  for unique connected subgroupoids  $\Gamma_i$ , for  $i \in I$ , and  $\Lambda_j$ , for  $j \in J$ , of  $\Gamma$  and  $\Lambda$  respectively. Therefore

$$\text{hom}(\Gamma, \Lambda) = \times_{i \in I, j \in J} \text{hom}(\Gamma_i, \Lambda_j)$$

and hence

$$|\mathrm{hom}(\Gamma, \Lambda)| = \prod_{i \in I, j \in J} |\mathrm{hom}(\Gamma_i, \Lambda_j)|.$$

Using Theorem 2 on each  $|\mathrm{hom}(\Gamma_i, \Lambda_j)|$  we can work out the cardinality of the set of elementary  $\Lambda$ -gradings on  $K[\Gamma]$ .

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